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The uniqueness of the exact solution of the Riemann problem for the shallow water equations with discontinuous bottom

Andrey I. Aleksyuk[a,b,*], Vitaly V. Belikov[b]

[a] *Faculty of Mechanics and Mathematics, Lomonosov Moscow State University, Moscow 119991, Russian Federation*
[b] *Water Problems Institute, Russian Academy of Sciences, Moscow 119333, Russian Federation*

Abstract

The Riemann problem for the shallow water equations with discontinuous topography is considered. In a general case the exact solution of this problem is not unique, which complicates the application of an exact Riemann solver in numerical methods, since it is not clear which solution should be chosen. In the present work it is shown that involving an additional physical assumption makes it possible to prove the existence and uniqueness of the solution. The assumption is that the discharge at the bottom discontinuity should continuously depend on the initial conditions. The proven uniqueness opens up a possibility to use an exact Riemann solver for a numerical solution of the shallow water equations with complex discontinuous topography.

Keywords: Riemann problem, shallow water equations, discontinuous bottom geometry, uniqueness of solutions,

1. Introduction

We consider a one-dimensional flow of a fluid in a gravitational field above a discontinuous bottom. It is assumed that the fluid motion is described by a system of the shallow water equations [1]

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial (hu)}{\partial x} = 0, \\ \dfrac{\partial (hu)}{\partial t} + \dfrac{\partial}{\partial x}\left(hu^2 + \dfrac{1}{2}gh^2\right) = -gh\dfrac{\partial b}{\partial x}. \end{cases} \tag{1}$$

Here $h(x,t)$ and $u(x,t)$ are the depth of the fluid and the depth-averaged horizontal component of velocity; g is the gravity constant; x and t are the coordinate in the horizontal direction and time. Function $b(x)$ determining the bottom topography without loss of generality can be given by relation

$$b(x) = \begin{cases} 0, & x < 0, \\ \Delta b, & x > 0. \end{cases} \tag{2}$$

We assume that Δb is a positive constant.

At the initial moment of time $t = 0$, to the left $(x < 0)$ and right $(x > 0)$ of the bottom discontinuity, the flow parameters are constant:

$$(h, u) = \begin{cases} (h_L, u_L), & x < 0, \\ (h_R, u_R), & x > 0. \end{cases} \tag{3}$$

This is the Riemann problem with bottom discontinuity, which was studied in many publications [2–16]. The interest in it can be explained by the fact that the problem arises as part of Godunov-type numerical methods for the shallow water equations when the bottom topography is approximated by a piecewise continuous function. Such methods are widespread for solving hyperbolic problems[17]. However, in the case of a discontinuous bottom, the application of the Riemann problem can be problematic due to the nonuniqueness of the solution. This paper presents a possible solution to this problem.

The peculiarity of the Riemann problem (1), (2), (3) lies in the fact that for $x = 0$ there is a bottom discontinuity and derivative $\partial b/\partial x$ in Eqs. (1) is not defined at $x = 0$. Hence, the classical consequence of the law of conservation of momentum does not hold: in general, $h_- u_-^2 + gh_-^2/2 \neq h_+ u_+^2 + gh_+^2/2$ (here, signs '-' and '+' denote values of parameters at $x \to -0$ and

*Corresponding author.
Email address: aleksyuk@mech.math.msu.su (Andrey I. Aleksyuk)

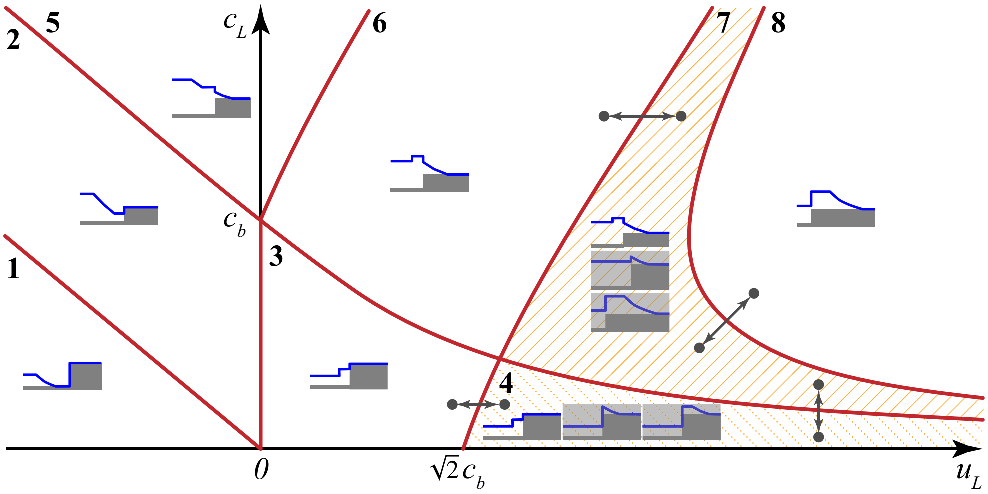


Fig. 1: Schematic representation of the flow regimes for the case with an initially dry bottom to the right of discontinuity. For initial conditions from domains 4 and 7 there are three possible solutions (shaded solutions do not satisfy the continuity condition). Here, $c_b = \sqrt{g\Delta b}$.

$x \to +0$). From the mass conservation law one can obtain relation $h_- u_- = h_+ u_+$, connecting parameters at $x = \pm 0$. The second relation can be obtained by treating the stationary bottom discontinuity as the limit of a monotonically changing continuous bottom as the length of the transition tends to zero (Section 4). The result of this procedure is three possible cases with different relations connecting parameters at $x = \pm 0$. Of particular interest here is the so-called resonant wave regime (see, for example,[6, 8]) when a stationary hydraulic jump appears at the bottom transition interval. For the initial conditions corresponding to the resonant regime the solution can be not unique. Another common approach for deriving relations at the bottom discontinuity is to consider it as an obstacle with a physically vertical wall (for example, various hydraulic structures), taking into account the reaction of the wall and/or dissipative processes, see, for example, [10–16].

Thus, in the present work the bottom discontinuity is physically considered as a monotonically changing continuous bottom, which in the shallow water approximation is represented as a discontinuity (in a sense, this is similar to the discontinuous representation of hydraulic jumps physically having finite length and internal structure). In a numerical simulation one can resolve this continuous bottom on a fine mesh and using the classical Riemann problem obtain the same result as for the Riemann problem with discontinuous bottom on a coarse mesh. However, it can be much less effective for practical problems with complex topography.

The solvability of the Riemann problem (1), (2), (3) was studied before for particular and arbitrary initial conditions [5, 6, 8–10]. The solution of this problem exists for arbitrary values of the initial conditions. However, depending on the initial conditions there can be up to three solutions.

The present paper shows for the first time that the Riemann problem has a unique solution if one additionally assumes that this solution should continuously depend on initial conditions in the following sense: *small changes of initial conditions cause small changes in discharge $q = hu$ at discontinuity ($x = 0$).* This idea for singling out a unique solution was suggested and applied in [9] in the particular case of the Riemann problem with $h_R = 0$. For greater clarity let us briefly repeat this result.

If at $x > 0$ the bottom is initially dry ($h_R = 0$) and Δb is fixed, the solution is determined by only two parameters, u_L and h_L (or $c_L = \sqrt{gh_L}$). The scheme for possible solutions in (u_L, c_L)-plane is shown in Fig. 1(see [9] for details). For initial conditions from each domain except 4 and 7 there is only one solution. Domains 4 and 7 correspond to the situation with three possible solutions (at the boundaries of domains 4 and 7 some of these solutions can become identical, therefore it is also possible that there are only two solutions). Let us check that solutions in each domain have continuously changing discharge at $x = 0$, when the initial conditions vary continuously between points in adjacent domains. Only one solution (not shaded in the figure) satisfies the transition between domains 6 and 7, since the other two solutions in domain 7 have the discharge ($h_L u_L$) different from the one in domain 6 (not equal to $h_L u_L$ behind the strong discontinuity moving to the left). This solution also satisfies the assumption for the boundary between domains 7 and 8, since at this boundary the strong discontinuity has zero velocity. Analogously, it can be demonstrated that one should consider only the solution that is not shaded in domain 4, as it satisfies the assumption for the transitions into domains 3 and 7.

In the present paper this idea is used for a general case with any initial conditions. It has been discovered, that as in the example considered it allows one to prove the uniqueness of the solution of the Riemann problem with discontinuous bottom. This opens up a possibility of using the exact Riemann solver as part of the numerical methods based on Godunov-type methods for the problems with a complex discontinuous topography (see examples of practical problems [18–20]).

We use an alternative proof of the solvability of the problem, which will be required to demonstrate the uniqueness of the solution. It should be noted that the key ideas are similar to the ones described in [2, 5, 6, 8, 21]. Some of the properties used were analyzed in earlier publications (mentioned above), nevertheless, for the integrity of the work the proof of certain auxiliary statements can be repeated.

In the next section, we describe the general steps in the method for solving the problem and the structure of the article.

2. Method for solving the problem

It is more convenient to study the problem in the nondimensional parameters

$$x' = \frac{x}{\Delta b}, \quad u' = \frac{u}{\sqrt{g\Delta b}}, \quad h' = \frac{h}{\Delta b}, \quad b' = \frac{b}{\Delta b}, \quad t' = t\sqrt{\frac{g}{\Delta b}}, \quad c' = \sqrt{h'}, \quad c'_b = \sqrt{b'}, \tag{4}$$

which allow one to eliminate the dependence of the solution on the parameter $\Delta b > 0$, as long as the dimensionless form of the Riemann problem (1), (2), (3) can be written in the following way (we omit primes here and further on)

$$\begin{cases} \dfrac{\partial(c^2)}{\partial t} + \dfrac{\partial(uc^2)}{\partial x} = 0, \\ \dfrac{\partial(c^2u)}{\partial t} + \dfrac{\partial}{\partial x}\left(c^2u^2 + \dfrac{1}{2}c^4\right) = -c^2\dfrac{\partial(c_b^2)}{\partial x}, \end{cases} \tag{5}$$

$$c_b(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0, \end{cases} \qquad \boldsymbol{w}(x,0) = \begin{cases} \boldsymbol{w_L}, & x < 0, \\ \boldsymbol{w_R}, & x > 0. \end{cases} \tag{6}$$

Here, the short notations are introduced: the solution of the problem is $\boldsymbol{w}(x,t) = (c,u)$; $\boldsymbol{w_L} = (c_L, u_L)$ and $\boldsymbol{w_R} = (c_R, u_R)$; additionally, the values of $\boldsymbol{w}$ at $x = -0$ and $x = +0$ will be denoted by $\boldsymbol{w_-} = (c_-, u_-)$ and $\boldsymbol{w_+} = (c_+, u_+)$.

The following steps are taken to show that there exists a unique solution of the problem (5), (6) for any initial conditions $\boldsymbol{w_L}$, $\boldsymbol{w_R}$.

(1) In Section 3, the flows at $x < 0$ and $x > 0$ are considered separately; all possible solutions in half-planes are sought in the form of a combination of elementary exact solutions: shock waves (bores or hydraulic jumps), rarefaction waves (simple waves) and the regions with constant parameters. If $\boldsymbol{w_L}$ ($\boldsymbol{w_R}$) is given, each solution in half-plane $x < 0$ ($x > 0$) can be uniquely defined by value $\boldsymbol{w_-}$ ($\boldsymbol{w_+}$). That is why all possible solutions are described by set $\mathcal{D}_-$ ($\mathcal{D}_+$) of admissible values of $\boldsymbol{w_-}$ ($\boldsymbol{w_+}$) that depend only on the given $\boldsymbol{w_L}$ ($\boldsymbol{w_R}$).

(2) In Section 4, the equations connecting parameters $\boldsymbol{w_-}$ and $\boldsymbol{w_+}$ are derived using the conservation laws (5) considering the discontinuity at $x = 0$ as the limiting case of a monotonically changing continuous bottom on an interval, whose length tends to zero.

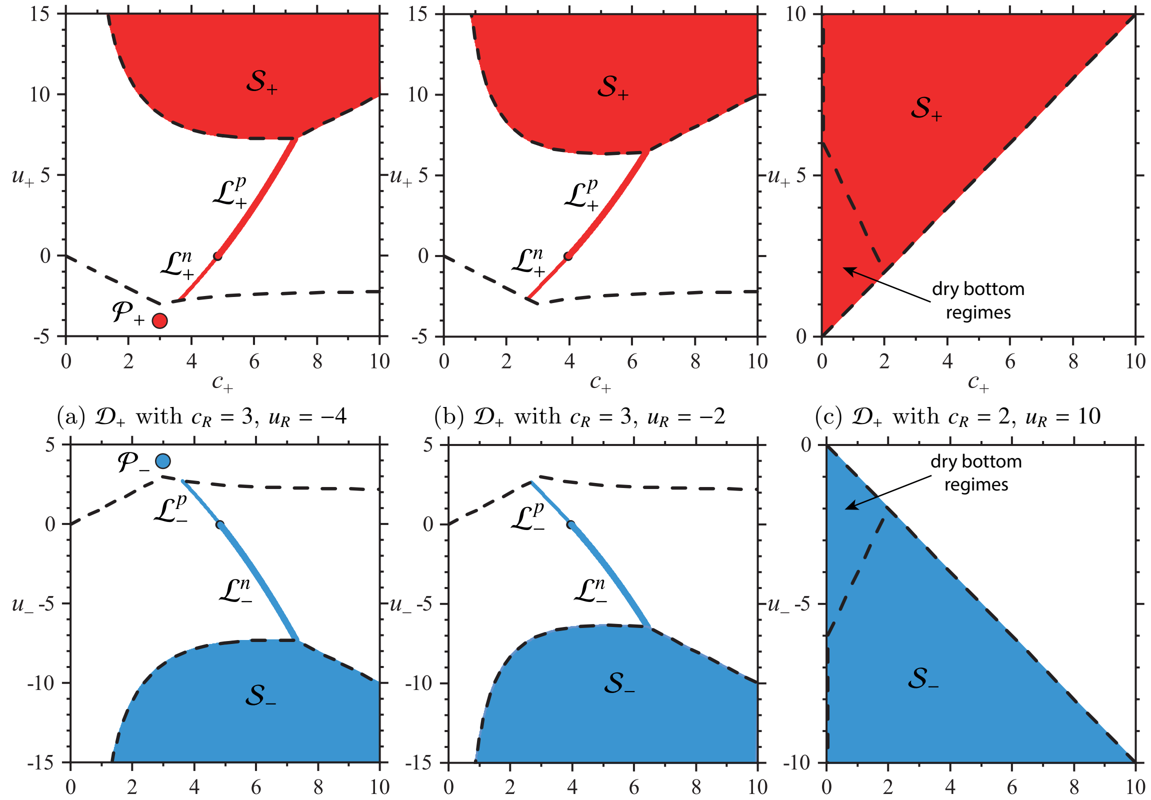
(3) In Section 5, the existence of the solution is studied separately for three possible situations: $q = 0$, $q > 0$, $q < 0$, where $q = u_-c_-^2 = u_+c_+^2$ is the discharge through the cross-section $x = 0$. For each of these three situations we construct the image of one of the sets $\mathcal{D}_-$ or $\mathcal{D}_+$ using relations at $x = 0$ (Section 4). The direction of the mapping (from $x = +0$ to $x = -0$ or in the opposite way) is selected along the flow direction (except when $q = 0$), which greatly facilitates finding an image. In the next step we look for intersections of the image of one of the sets $\mathcal{D}_-$ or $\mathcal{D}_+$ with the other set. There can be up to three intersection points. Every such intersection corresponds to the solution of the problem (5), (6). However, not all of these solutions satisfy the condition of continuous dependence of q on the initial conditions.

(4) In Section 6, Theorem 1 on the existence and uniqueness of the solution is formulated under the assumption that q continuously depends on the initial conditions. It is shown that in the case when there are three solutions, two of them can be discarded. The approach used here is checking discharge q continuity at the bifurcation points, as it was suggested in [9] in the particular case of the Riemann problem with an initially dry bed at $x > 0$.

The examples of all possible configurations of the flow with $\Delta b > 0$ are given in Section 7.

3. The solution in the half-plane

Let us consider all possible solutions in the right half-plane $x \geq 0$ for a given $\boldsymbol{w_R}$. Each solution is represented as a combination of elementary exact solutions: shock waves, rarefaction waves (simple waves) and regions with constant parameters. As it is often done[1, 5, 8], here, we refer to a bore (or a hydraulic jump) and simple waves in shallow water as shock and rarefaction waves (or just as a shock and rarefaction), as in the terminology of gas dynamics. Two types of



(a) $\mathcal{D}_+$ with $c_R = 3$, $u_R = -4$ (b) $\mathcal{D}_+$ with $c_R = 3$, $u_R = -2$ (c) $\mathcal{D}_+$ with $c_R = 2$, $u_R = 10$

(d) $\mathcal{D}_-$ with $c_L = 3$, $u_L = 4$ (e) $\mathcal{D}_-$ with $c_L = 3$, $u_L = 2$ (f) $\mathcal{D}_-$ with $c_L = 2$, $u_L = -10$

Fig. 2: A characteristic example of sets $\mathcal{D}_+$ (a-c) and $\mathcal{D}_-$ (d-f). (a), (d) — $Fr_R < -1$ and $Fr_L > 1$, all sets are not empty; (b), (e) — $Fr_R > -1$ and $Fr_L < 1$, $\mathcal{P}_+ = \varnothing$ and $\mathcal{P}_- = \varnothing$; (c), (f) — $Fr_R > 2$ and $Fr_L < -2$, $\mathcal{P}_+ \cup \mathcal{L}_+ = \varnothing$ and $\mathcal{P}_- \cup \mathcal{L}_- = \varnothing$. The dashed lines represent the restrictions of Eqs. (9).

shock and rarefaction waves can be distinguished: 1- (left) and 2- (right) shock and rarefaction waves. k-wave ($k = 1, 2$), which is either k-shock or k-rarefaction, connects the parameters to the left (c_l, u_l) and to the right (c_r, u_r) of it by relations:

$$\begin{aligned} &\text{1-wave: } u_l = u_r + f(c_r, c_l), \\ &\text{2-wave: } u_l = u_r + f(c_l, c_r), \end{aligned} \qquad f(c_l, c_r) = \begin{cases} 2(c_l - c_r), & c_l \le c_r, \\ \frac{c_l^2 - c_r^2}{c_l c_r}\sqrt{\frac{1}{2}(c_l^2 + c_r^2)}, & c_l > c_r. \end{cases} \tag{7}$$

The solution of the classical Riemann problem consists of no more than one 1-wave, one 2-wave and one intermediate state (c_0, u_0). Thus, values $\boldsymbol{w_+}$ and $\boldsymbol{w_R}$ are related by the equations

$$u_+ = u_0 + f(c_0, c_+), \qquad u_0 = u_R + f(c_0, c_R), \tag{8}$$

Here, the bottom is not dry, i.e. $c_+ \ne 0$, $c_0 \ne 0$, $c_R \ne 0$. If $c_+ = c_0$ or $c_R = c_0$, then there is no 1-wave or 2-wave, respectively. The details on the k-waves terminology and the classical Riemann problem solution for the shallow water equations can be found, for example, in [5, 8, 22].

Due to the fact that shock and rarefaction waves are located at $x \ge 0$, the following restrictions on the possible values of $\boldsymbol{w_+}$ are imposed (see Appendix A).

$$\begin{aligned} &u_+ \ge \max(c_+, \sigma(c_0, c_+)), \text{ if } c_+ \ne c_0, \\ &u_0 \ge -\min(c_0, \sigma(c_R, c_0)), \text{ if } c_R \ne c_0. \end{aligned} \tag{9}$$

Here, $\sigma(c_1, c_2) = (c_1/c_2)\sqrt{(c_1^2 + c_2^2)/2}$.

Let $\mathcal{D}_+^1$ be the set of (c_+, u_+), that satisfies Eqs. (8) and (9) provided that there are no regions with a dry bottom.

Additionally, three situations are possible when a dry bottom can occur. The first one is $c = 0$ for any $x > 0$, hence $c_+ = 0$ and u_+ is arbitrary. The second is $c_R = 0$ and $c_+ > 0$. Then at $x > 0$ there should be 1-rarefaction wave, hence $\boldsymbol{w_+}$ can take any values under the condition $u_+ \ge c_+ > 0$. The third situation is $c_R \ne 0$ and $u_R \ge 2c_R$, where the second inequality is necessary for the existence of a dry bed region at $x > 0$. Then either $c_+ = 0$ with arbitrary value of u_+ or $\boldsymbol{w_+}$ can take any values provided that $0 < c_+ \le u_+ \le u_R - 2(c_R + c_+)$. These inequalities ensure the existence of an intermediate region with a dry bottom with rarefaction waves on either side of it.

The set of possible values of $\boldsymbol{w_+}$ with dry bottom regions is

$$\begin{aligned}\mathcal{D}_+^0 = \{(u_+, c_+):\ u_+ \geq c_+ > 0, \text{ if } c_R = 0\} \cup \{(u_+, c_+):\ 0 < c_+ \leq u_+ \leq u_R - 2(c_R + c_+), \text{ if } c_R \neq 0\} \\ \cup \{(u_+, c_+):\ c_+ = 0 \text{ and } u_+ \text{ is arbitrary}\}.\end{aligned} \tag{10}$$

Therefore, the set of the corresponding values of $\boldsymbol{w_+}$ is defined by the following relation:

$$\mathcal{D}_+ = \mathcal{D}_+^1 \cup \mathcal{D}_+^0. \tag{11}$$

It can be seen that if $c_R = 0$ or if $c_R \neq 0$ and $u_R \geq 2c_R$, then set $\mathcal{D}_+$ is simply any $\boldsymbol{w_+}$ such that $u_+ \geq c_+ \geq 0$ (see, for example, Fig. 2c).

Examples of set $\mathcal{D}_+$ for a given $\boldsymbol{w_R}$ are shown in Fig. 2a-c. It is convenient to represent $\mathcal{D}_+$ in the form of a union of three disjoint subsets, $\mathcal{D}_+ = \mathcal{P}_+ \cup \mathcal{L}_+ \cup \mathcal{S}_+$:

- $\mathcal{P}_+$ consists of one point (u_R, c_R) if $u_R < -c_R$, otherwise the set is empty;
- $\mathcal{L}_+ = \{(u_+, c_+)\ :\ u_+ = u_R + f(c_+, c_R),\ u_+ < c_+,\ u_+ \geq -\min(c_+, \sigma(c_R, c_+))\}$. Set $\mathcal{L}_+$ is not empty at $u_R < 2c_R$. $\mathcal{L}_+$ is represented as a union of subsets with $u_+ \geq 0$ ($\mathcal{L}_+^p$) and $u_+ < 0$ ($\mathcal{L}_+^n$);
- $\mathcal{S}_+ = \mathcal{D}_+ \cap \{u_+ \geq c_+\}$. If $u_R \geq 2c_R$ or $c_R = 0$, then $\mathcal{S}_+$ is the set of points defined by the inequalities $u_+ \geq c_+ \geq 0$.

These subsets also divide $\mathcal{D}_+$ by Froude number:

$$\begin{aligned} Fr_+ &< -1, && \text{if } \boldsymbol{w_+} \in \mathcal{P}_+; \\ -1 \leq Fr_+ &< 0, && \text{if } \boldsymbol{w_+} \in \mathcal{L}_+^n; \\ 0 \leq Fr_+ &< 1, && \text{if } \boldsymbol{w_+} \in \mathcal{L}_+^p; \\ 1 &\leq Fr_+, && \text{if } \boldsymbol{w_+} \in \mathcal{S}_+. \end{aligned} \tag{12}$$

Obviously with a given $\boldsymbol{w_R}$ each $\boldsymbol{w_+} \in \mathcal{D}_+$ corresponds uniquely to some configuration of the flow at $x \geq 0$ (otherwise, the classical Riemann problem with some $\boldsymbol{w_L}$, $\boldsymbol{w_R}$ does not have a unique solution, which is not true). Formally, if at some interval the depth is zero in shallow water equations (see Eqs. (1)), then the velocity is arbitrary, however, such solutions are not considered as different based on the physical meaning.

Set $\mathcal{D}_-$ of all possible values $\boldsymbol{w_-}$ to the left of the discontinuity is described in a similar way (Fig. 2d-f).

An important property of set $\mathcal{D}_+$ ($\mathcal{D}_-$) is that when the flow is moving from right to left, $u_+ < 0$ (from left to right, $u_- > 0$), the set of admissible values $\boldsymbol{w_+}$ ($\boldsymbol{w_-}$) is a fragment of a strictly monotonic curve $\mathcal{L}_+^n$ ($\mathcal{L}_-^p$) (see Appendix B, Lemma B-3) and a point $\mathcal{P}_+$ ($\mathcal{P}_-$) (see Fig. 2). In the following this circumstance will make it possible to simplify the stage of mapping sets $\mathcal{D}_+$, $\mathcal{D}_-$ through the bottom discontinuity: if it is carried out along the flow direction, then there is no need to map complex sets $\mathcal{S}_+$, $\mathcal{S}_-$.

4. Conservation laws for a bottom discontinuity

The equations connecting the flow parameters at $x = -0$ and $x = +0$ are obtained from the consideration of the bottom discontinuity as a limiting case of monotonically changing continuous bottom function $b(x)$. Let us consider a stationary flow in the interval $(-\varepsilon, \varepsilon)$, for which the bottom function $b(x)$ continuously increases from $b(-\varepsilon) = 0$ to $b(\varepsilon) = 1$. There are three possibilities with different conditions at $x = 0$ as $\varepsilon \to 0$.

I. The bottom is partially or completely dry. Therefore, as $\varepsilon \to 0$ we have $c_+ = 0$, u_+ can be arbitrary, $u_- = 0$, and c_- can be arbitrary, provided that $c_- \leq 1$ (if $c_- = 0$, u_- can be arbitrary).

II. The bottom is not dry and there is no shock wave inside the interval $(-\varepsilon, \varepsilon)$. From Eqs. (5), provided that $c \neq 0$,

$$\begin{cases} \dfrac{d(uc^2)}{dx} = 0, \\ \dfrac{d}{dx}\left(\dfrac{1}{2}u^2 + c^2 + c_b^2\right) = 0. \end{cases} \tag{13}$$

Integration of Eqs. (13) from $-\varepsilon$ to ε in the limit as $\varepsilon \to 0$ gives the following conditions

$$\begin{aligned} &\frac{1}{2}u_-^2 + c_-^2 = \frac{1}{2}u_+^2 + c_+^2 + 1, \\ &u_- c_-^2 = u_+ c_+^2. \end{aligned} \tag{14}$$

It should be noted that there is a particular solution of this system, which corresponds to the equilibrium state with a horizontal free surface: $u_- = u_+ = 0$ and $c_-^2 = c_+^2 + 1$. The limit of such a solution as $c_+ \to 0$ is the equilibrium state from case I.

Eqs. (14) can be rewritten in the form

$$\begin{aligned} &\varphi(u_-) = \varphi(u_+) + 1, \\ &\varphi(u) = \frac{1}{2}u^2 + \frac{u_*^3}{u}, \\ &u_*^3 = q = u_- c_-^2 = u_+ c_+^2. \end{aligned} \tag{15}$$

The existence and uniqueness properties for solution of these equations with given $\boldsymbol{w_-}$ or $\boldsymbol{w_+}$ are considered in Appendix B. The main conclusions are as follows.

If $\boldsymbol{w_-}$ is known and the solution of Eqs. (15) exists, then there is only one solution which we denote as $\boldsymbol{w_+} = \Phi_{LR}(\boldsymbol{w_-})$. The condition for the existence of the solution is $\varphi(u_-) \geq \varphi(u_*) + 1$.

If $\boldsymbol{w_+}$ is known and $Fr_+ = u_+/c_+ \neq \pm 1$, then there exists a unique solution of Eqs. (15) and we denote it as $\boldsymbol{w_-} = \Phi_{RL}(\boldsymbol{w_+})$. However, if $Fr_+ = u_+/c_+ = \pm 1$, then image $\Phi_{RL}(\boldsymbol{w_+})$ contains two elements: one with $|Fr_-| > 1$ ($Fr_- = u_-/c_-$), and the other with $|Fr_-| < 1$.

III. The bottom is not dry and there is a shock wave inside the interval $x \in (-\varepsilon, \varepsilon)$. This case is the so-called resonant wave (see, for example, [21]). To the left and to the right of the shock wave, the flow is continuous and the values at the shock can be determined using the considerations of case II. We introduce an intermediate bottom level Δb^*, $0 < c_b^* = \sqrt{\Delta b^*} < 1$, which corresponds to the position of the shock, and values $\boldsymbol{w_-^*}, \boldsymbol{w_+^*}$ to the left and right of it. These values are related by the jump conditions for the shock with zero propagation velocity. Behind and before the shock we have a subcritical and supercritical flow, respectively. Hence, three equations connect $\boldsymbol{w_-}$ and $\boldsymbol{w_+}$,

$$\begin{aligned} &\varphi(u_-) = \varphi(u_-^*) + c_b^{*2}, \\ &\varphi(u_+^*) = \varphi(u_+) + 1 - c_b^{*2}, \\ &\psi(u_-^*) = \psi(u_+^*). \end{aligned} \tag{16}$$

Here, the third relation is the consequence of the jump conditions, $\psi(u) = u_* u + u_*^4/(2u^2)$. Since the flow is stationary, discharge q (and u_*) remains constant.

For given $\boldsymbol{w_-}$ and $\boldsymbol{w_+}$ we introduce the notations for mappings (multivalued functions) defined by Eqs. (16): $\{\boldsymbol{w_+}\} = \Psi_{LR}(\boldsymbol{w_-})$ and $\{\boldsymbol{w_-}\} = \Psi_{RL}(\boldsymbol{w_+})$. If image $\Psi_{LR}(\boldsymbol{w_-})$ or $\Psi_{RL}(\boldsymbol{w_+})$ exists, then it corresponds to a strictly monotonic function $u_+(c_+)$ or $u_-(c_-)$ (see Appendix C, Lemma C-4).

It should be noted that the limit configurations at $c_b^* \to 0$ and $c_b^* \to 1$ are physically equal to II-case configurations, because for both cases there is a stationary shock either at $x = -0$ or at $x = +0$, i.e. outside the interval $x \in (-\varepsilon, \varepsilon)$.

It is important that one cannot omit case III, otherwise, for some values of $\boldsymbol{w_L}, \boldsymbol{w_R}$, the solution of the Riemann problem does not exist (see, for example, Fig. 5i, j in Section 7).

5. The existence of a solution

Let us point out that for a given $\boldsymbol{w_L}$ ($\boldsymbol{w_R}$) the set $\mathcal{D}_-$ ($\mathcal{D}_+$) of possible values $\boldsymbol{w_-}$ ($\boldsymbol{w_+}$) cannot be empty, see Section 3 (otherwise, it would mean that the solution of the classical Riemann problem does not exist for some initial conditions). Therefore, the investigation of the solvability of the Riemann problem (5), (6) is reduced to the study of the opportunities to connect $\boldsymbol{w_-} \in \mathcal{D}_-$ and $\boldsymbol{w_+} \in \mathcal{D}_+$ using the relations of cases I, II, II from Section 4. In the following subsections we separately consider flows with zero, positive and negative discharge q at $x = 0$.

5.1. *Case A: no flow,* $q = 0$

In Section 4 it is shown that $q = 0$ could occur with different conditions connecting $\boldsymbol{w_-}$ and $\boldsymbol{w_+}$: I and II. These two situations can be expressed as follows.

A1. $c_+ = 0$, $c_- \leq 1$, $u_- = 0$ (if $c_- = 0$, u_- can be arbitrary). Equality $c_+ = 0$ imposes restrictions on values $\boldsymbol{w_R}$: $u_R \geq 2c_R$ or $c_R = 0$. Analogously, if $c_- = 0$, then $u_L \leq -2c_L$ or $c_L = 0$. If $0 < c_- \leq 1$, then value $\boldsymbol{w_-} \in \mathcal{L}_-^n$ and $u_- = 0$, hence, $u_L = f(c_-, c_L)$.

A2. $c_+ \neq 0$, $c_-^2 = c_+^2 + 1$ and $u_- = u_+ = 0$. These points belong to lines $\mathcal{L}_+, \mathcal{L}_-$ and can be considered as the limiting case ($q \to 0$) of the situations $q > 0$, $q < 0$ described in the following subsections.

The following lemma shows that these cases cannot be simultaneously true for given initial conditions.

Lemma 1. *If for a given $\boldsymbol{w_L}$, $\boldsymbol{w_R}$ the solution with $q = 0$ exists, it is unique among the solutions with $q = 0$.*

Proof. One can notice that if $u_R \geq 2c_R$ or $c_R = 0$ then $q = 0$ only if $c_+ = 0$. Hence, with given $\boldsymbol{w_L}$, $\boldsymbol{w_R}$ two solutions with different types (A1 and A2) are not possible, because for the solutions of type A1 either $u_R \geq 2c_R$ or $c_R = 0$ and for the solutions of type A2 $c_+ \neq 0$.

There cannot be several solutions of type A1. Indeed, there is a unique configuration to the right with $c_+ = 0$ and given $\boldsymbol{w_R}$ (otherwise, the classical Riemann problem with some $\boldsymbol{w_L}$, $\boldsymbol{w_R}$ does not have a unique solution, which is not true). Also, there is a unique configuration to the left with $u_- = 0$ or $c_- = 0$ and given $\boldsymbol{w_L}$ (see Fig. 2d-f).

Similarly, there cannot be more than one solution of type A2. □

5.2. Case B: flow from left to right, $q > 0$

The mapping of set $\mathcal{D}_-$ of admissible values of $\boldsymbol{w}_-$ using Eqs. (15), (16) connecting $\boldsymbol{w}_-$ and $\boldsymbol{w}_+$ can be written as $\Phi_{LR}(\mathcal{D}_-) \cup \Psi_{LR}(\mathcal{D}_-)$. The solution of the Riemann problem (5), (6) exists if the image has an intersection with set $\mathcal{D}_+$:

$$[\Phi_{LR}(\mathcal{D}_-) \cup \Psi_{LR}(\mathcal{D}_-)] \cap \mathcal{D}_+ \neq \varnothing. \tag{17}$$

Using the decomposition of sets $\mathcal{D}_-$, $\mathcal{D}_+$ (Section 3) and taking into account the fact that $q > 0$ (mappings Φ_{LR} and Ψ_{LR} do not change the sign of q) we obtain a simplified condition

$$\left[\Phi_{LR}(\mathcal{P}_- \cup \mathcal{L}_-^p) \cup \Psi_{LR}(\mathcal{P}_- \cup \mathcal{L}_-^p)\right] \cap [\mathcal{L}_+^p \cup \mathcal{S}_+] \neq \varnothing. \tag{18}$$

There are only five types of intersections:

- B1. $\Phi_{LR}(\mathcal{P}_-) \cap \mathcal{S}_+ \neq \varnothing$ $(Fr_+ \geq 1)$;
- B2. $\Phi_{LR}(\mathcal{L}_-^p) \cap \mathcal{S}_+ \neq \varnothing$ $(Fr_+ = 1)$;
- B3. $\Phi_{LR}(\mathcal{L}_-^p) \cap \mathcal{L}_+^p \neq \varnothing$ $(Fr_+ < 1)$;
- B4. $\Psi_{LR}(\mathcal{P}_-) \cap \mathcal{S}_+ \neq \varnothing$ $(Fr_+ = 1)$;
- B5. $\Psi_{LR}(\mathcal{P}_-) \cap \mathcal{L}_+^p \neq \varnothing$ $(Fr_+ < 1)$.

Considering the inequalities for the Froude number Fr_+ for elements from each pair of subsets it can be shown that other intersections are not possible (see Eqs. (12) and Appendix B, Lemma B-1, Appendix C , Lemma C-1). For instance, $\Phi_{LR}(\mathcal{P}_-) \cap \mathcal{L}_+^p = \varnothing$ because $Fr_- > 1$ for $\boldsymbol{w}_- \in \mathcal{P}_-$, hence, the image $\boldsymbol{w}_+ \in \Phi_{LR}(\mathcal{P}_-)$ has $Fr_+ \geq 1$ (see Appendix B, Lemma B-1) and it cannot belong to $\mathcal{L}_+^p$ with $Fr_+ < 1$.

In the next two lemmas we consider the question of the coexistence of solutions B1-B5.

Lemma 2. *In each case B1-B5, if the intersection is non-empty, then it contains only one point.*

Proof. Let us consider each case.

B1: $\mathcal{P}_-$ can contain only one point and function $\Phi_{LR}(\mathcal{P}_-)$ is single-valued (see Appendix B, Lemma B-1).

B2, B4: $Fr_+ \geq 1$ for points of $\mathcal{S}_+$. There can be only one point from $\Phi_{LR}(\mathcal{L}_-^p)$ as well as from $\Psi_{LR}(\mathcal{P}_-)$ with $Fr_+ = 1$, for others $Fr_+ < 1$ (see Appendix B, Lemmas B-1, B-4 and Appendix C, Lemmas C-1, C-4).

B3, B5: Lines $\mathcal{L}_+^p$, $\Phi_{LR}(\mathcal{L}_-^p)$ and $\Psi_{LR}(\mathcal{P}_-)$ can be seen as functions $u_+ = u_+(c_+)$. The first one of them strictly increases, the other two strictly decrease (see Appendix B, Lemmas B-3, B-4 and Appendix C, Lemma C-4). Therefore if the intersection between the first and the second lines as well as between the first and the third lines exists, this intersection is the only one. □

Lemma 3. *Only one of four situations is possible:*

(1) none of the conditions B1-B5 is satisfied;

(2) only one condition is satisfied: B1, B2 or B3;

(3) only two conditions are satisfied: B1, B2 or B1, B3 or B1, B4 or B1, B5;

(4) only three conditions are satisfied: B1, B2, B4 or B1, B3, B4 or B1, B3, B5.

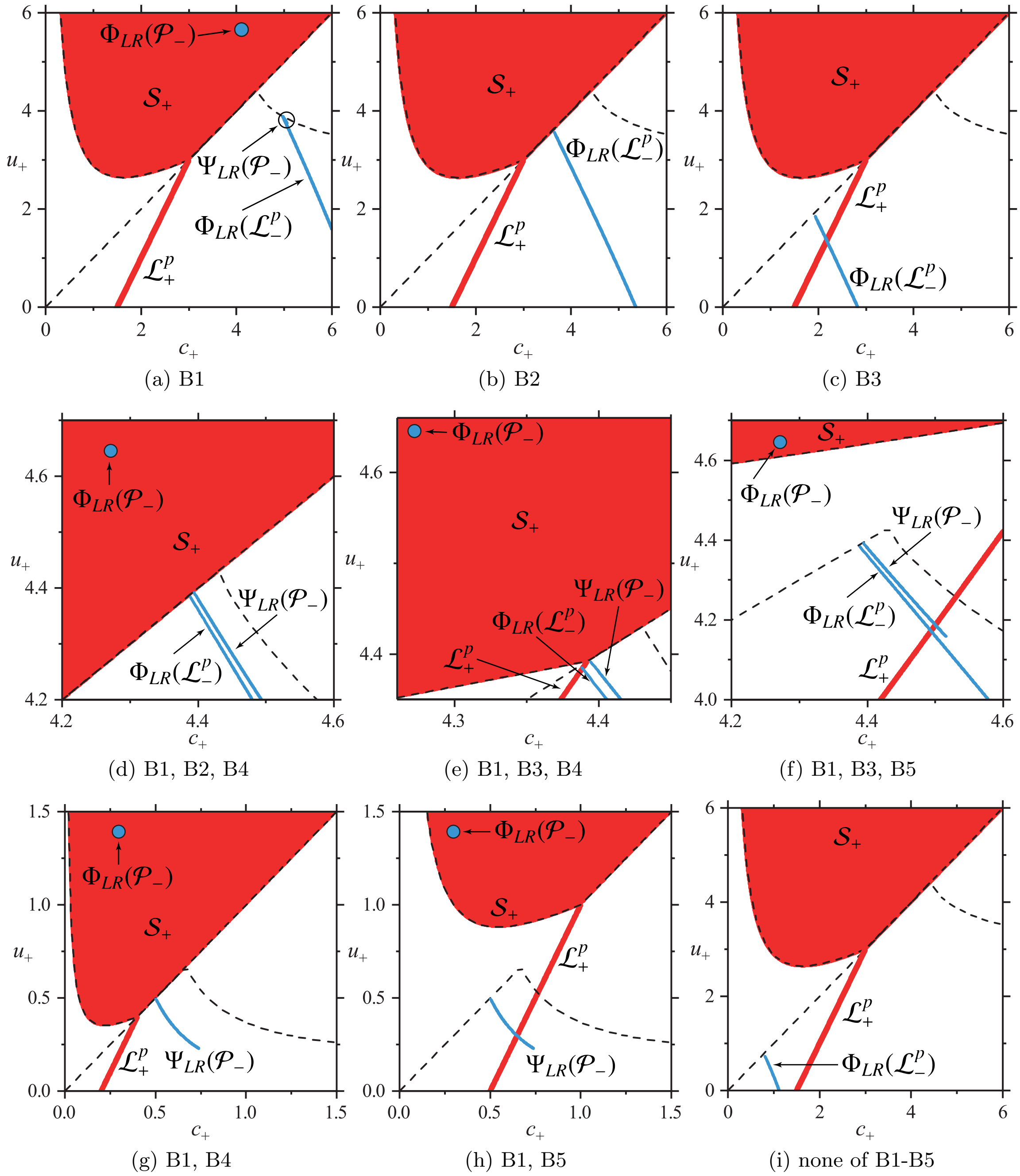


Fig. 3: Possible configurations of type B ($q > 0$). Initial conditions: (a)-(c) — $\boldsymbol{w}_L = (6,4), (3,4), (-2,4)$ and $\boldsymbol{w}_R = (3,3)$; (d)-(f) — $\boldsymbol{w}_L = (5.3,4)$ and $\boldsymbol{w}_R = (3,3), (1.455,3), (1,3)$; (g) — $\boldsymbol{w}_L = (2,0.25)$, $\boldsymbol{w}_R = (0.4,0.4)$; (h) — $\boldsymbol{w}_L = (2,0.25)$, $\boldsymbol{w}_R = (1,1)$; (i) — $\boldsymbol{w}_L = (-5,4)$, $\boldsymbol{w}_R = (3,3)$. Here, two intermediate configurations B1, B2 and B1, B3 are not shown. These configurations correspond to plots (d)-(f), when point $\Phi_{LR}(\mathcal{P}_-)$ belongs to the border of $\mathcal{S}_+$ and $\Psi_{LR}(\mathcal{P}_-) = \varnothing$. The dashed lines represent the restrictions of Eqs. (9).

Proof. First of all, one can see that each of the situations is possible by looking into examples in Fig. 3. Now let us show that the other combinations are not allowed.

Cases B2 and B3 cannot be simultaneously true. This can be seen in Fig. 3. Lines $\mathcal{L}_+^p$ and $\mathcal{S}_+ \cap \{Fr_+ = 1\}$ can be seen as one strictly increasing curve $u_+ = u_+(c_+)$ (see Appendix B, Lemma B-3). At the same time line $\Phi_{LR}(\mathcal{L}_-^p)$ represents a strictly decreasing curve $u_+ = u_+(c_+)$ (see Appendix B, Lemma B-4). Hence, they can have only one intersection. Similarly, *conditions B4 and B5 cannot be simultaneously true.*

If B4 or B5 is true, then B1 is also true. Indeed, let B4 or B5 be true, $\boldsymbol{w}_- \in \mathcal{P}_-$ ($Fr_- > 1$), the intersection point is denoted by $\boldsymbol{w}_+$, i.e. $\boldsymbol{w}_+ \in \Psi_{LR}(\mathcal{P}_-) \cap \mathcal{S}_+$ or $\boldsymbol{w}_+ \in \Psi_{LR}(\mathcal{P}_-) \cap \mathcal{L}_+^p$. $\Phi_{LR}(\mathcal{P}_-)$ is not empty, since the requirements for the existence of the solution of Eqs. (15) (see Appendix B, Lemma B-1) is less stringent than for the existence of a resonant wave (see Appendix C, Lemma C-3. Let us show that $\boldsymbol{w}'_+ = \Phi_{LR}(\boldsymbol{w}_-)$ belongs to $\mathcal{S}_+$. Note that $Fr'_+ \geq 1$ ($Fr'_+ = u'_+/c'_+$), since $Fr_- > 1$. Therefore, it is only necessary to verify that point $\boldsymbol{w}'_+$ lies higher than the left curve bounding the set $\mathcal{S}_+$ (see Eqs. (9) and Fig. 3). This curve is defined by the restriction that if in the right half-plane there is a 1-shock wave, then its propagation velocity is greater than or equal to zero. If there is no 1-shock wave, $\boldsymbol{w}'_+ \in \mathcal{S}_+$, since the solution satisfies other restrictions and can be connected to $\boldsymbol{w_R}$ using Eq. (8). Let us show that if there is a 1-shock wave (in the solution corresponding to $\boldsymbol{w}'_+$), it has nonnegative velocity.

One can notice, that discharge q at $x = 0$ is the same for $\boldsymbol{w_+}$ and $\boldsymbol{w}'_+$, because $\boldsymbol{w_-}$ is the same. Let us consider a point $\boldsymbol{w^0_+}$, which has the same discharge and lies on the border of $\mathcal{S}_+$, i.e. $\psi(u^0_+) = \psi(u_+)$. Hence, if $u'_+ \geq u^0_+$, then $\boldsymbol{w}'_+ \in \mathcal{S}_+$ (1-shock wave has nonnegative velocity). The fact that $u'_+ \geq u^0_+$ can be shown as follows. We have the system of equations

$$\begin{aligned} &\varphi(u_-) = \varphi(u'_+) + 1, \\ &\varphi(u_-) = \varphi(u^*_-) + c_b^{*2}, \\ &\varphi(u^*_+) = \varphi(u_+) + 1 - c_b^{*2}, \\ &\psi(u^*_-) = \psi(u^*_+), \\ &\psi(u^0_+) = \psi(u_+). \end{aligned} \tag{19}$$

From these relations it follows that $u'_+ \leq u^*_-$ (the first and second relations), $u^*_+ \leq u_+$ (the third relation) and

$$\psi(u^0_+) - \psi(u'_+) = [\chi(u'_+) - \chi(u^*_-)] + [\chi(u^*_+) - \chi(u_+)], \tag{20}$$

here $\chi(u) = \varphi(u) - \psi(u)$ is an increasing function. Therefore, $\psi(u^0_+) - \psi(u'_+) \leq 0$ and, consequently, $u^0_+ \leq u'_+$, since $Fr^0_+ \geq 1$, $Fr'_+ \geq 1$ ($Fr^0_+ = u^0_+/c^0_+$).

Finally, let us show that *three-point intersection B1, B2, B5 is not possible.* This is the consequence of the fact that the curve for $\Psi_{LR}(\mathcal{P}_-)$ lies no lower than the curve for $\Phi_{LR}(\mathcal{L}^p_-)$ (the intersection is possible at the leftmost point, when $c^*_b = 0$), see Fig. 3. Indeed, $\mathcal{P}_- \neq \varnothing$, so $\boldsymbol{w_-}$ is either equal to $\boldsymbol{w_L}$, or connected to $\boldsymbol{w_L}$ by the 1-shock wave. This means that $q \leq u_L c_L^2$ for $\boldsymbol{w_+} \in \Phi_{LR}(\boldsymbol{w_-})$ (velocity of 1-shock wave is nonpositive), and for resonant wave values $u_L c_L^2$ and q coincide, since $\boldsymbol{w_-} = \boldsymbol{w_L}$. That is why the curve for $\Psi_{LR}(\mathcal{P}_-)$ is no lower than the curve for $\Phi_{LR}(\mathcal{L}^p_-)$. Further, the curves $\mathcal{L}^p_+$ and $\mathcal{S}_+$ for $Fr_+ = 1$ can be considered as one increasing curve $u_+(c_+)$ (see Fig. 2a-c), and $\mathcal{L}^p_+$ is the lower part of this combined curve. Hence, if there is intersection B5 ($\Psi_{LR}(\mathcal{P}_-)$ and $\mathcal{L}^p_+$), then intersection B2 ($\Phi_{LR}(\mathcal{L}^p_-)$ and $\mathcal{S}_+$ for $Fr_+ = 1$) is not possible. □

5.3. Case C: flow from right to left, $q < 0$

Similarly to the previous subsection, the solution exists if

$$[\Phi_{RL}(\mathcal{D}_+) \cup \Psi_{RL}(\mathcal{D}_+)] \cap \mathcal{D}_- \neq \varnothing. \tag{21}$$

Using decomposition of sets $\mathcal{D}_-$, $\mathcal{D}_+$ and taking into account the fact that $q < 0$ (mappings Φ_{RL} and Ψ_{RL} do not change the sign of q) we obtain a simplified condition

$$[\Phi_{RL}(\mathcal{P}_+ \cup \mathcal{L}^n_+) \cup \Psi_{RL}(\mathcal{P}_+ \cup \mathcal{L}^n_+)] \cap [\mathcal{L}^n_- \cup \mathcal{S}_-] \neq \varnothing. \tag{22}$$

Considering the inequalities for the Froude number (and conditions for the existence of the mappings) one can show that only five types of intersections are possible (see Eqs. (12) and Appendix B, Lemma B-2, Appendix C, Lemma C-1):

C1. $\Phi_{RL}(\mathcal{P}_+) \cap \mathcal{S}_- \neq \varnothing$ ($Fr_- < -1$);

C2. $\Phi_{RL}(\mathcal{L}^n_+) \cap \mathcal{S}_- \neq \varnothing$ ($Fr_- < -1$, $Fr_+ = -1$);

C3. $\Phi_{RL}(\mathcal{L}^n_+) \cap \mathcal{L}^n_- \neq \varnothing$ ($Fr_- > -1$);

C4. $\Psi_{RL}(\mathcal{P}_+) \cap \mathcal{L}^n_- \neq \varnothing$ ($Fr_- > -1$);

C5. $\Psi_{RL}(\mathcal{L}^n_+) \cap \mathcal{L}^n_- \neq \varnothing$ ($Fr_- > -1$, $Fr_+ = -1$).

In the next two lemmas we consider the question of the coexistence of solutions C1-C5.

Lemma 4. *In each case C1-C4, if the intersection is non-empty, then it contains only one point.*

Proof. C1: $\mathcal{P}_+$ contains only one point with $Fr_+ < -1$ and function $\Phi_{RL}(\mathcal{P}_+)$ is single-valued if $Fr_+ < -1$ (see Appendix B, Lemma B-2).

C2: $\mathcal{L}^n_+$ and $\mathcal{S}_-$ contain points with $Fr_+ \geq -1$ and $Fr_- \leq -1$. The intersection is not empty only when $Fr_+ = -1$ and, hence, the image of Φ_{RL} has only one point with $Fr_- < -1$ (the second point has $Fr_- > -1$, see Appendix B, Lemma B-2).

C3-C5: Lines $\mathcal{L}^n_-$, $\Phi_{RL}(\mathcal{L}^n_+)$, $\Psi_{RL}(\mathcal{P}_+)$ and $\Psi_{RL}(\mathcal{L}^n_+)$ (this one exists only for one point in $\mathcal{L}^n_+$ with $Fr_+ = -1$) can be seen as functions $u_- = u_-(c_-)$. The first one of them strictly decreases, the other three strictly increase (see Appendix B, Lemmas B-3, B-4 and Appendix C, Lemma C-4). Therefore if the intersection between the first and the second or third or forth lines exists, this intersection is unique. □

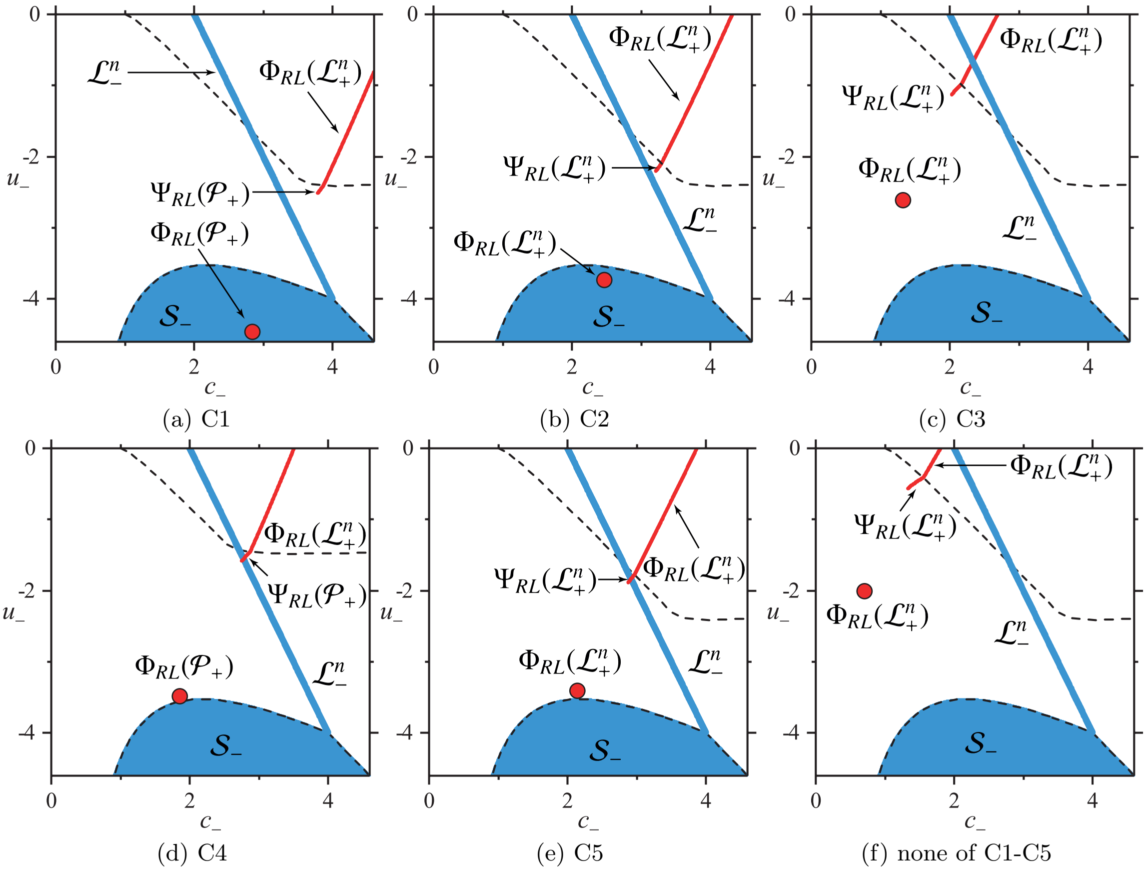


Fig. 4: Possible configurations of type C ($q < 0$). Initial conditions: (a)-(f) — $\boldsymbol{w}_L = (-4, 4)$ and $\boldsymbol{w}_R = (-4, 3)$, $(-2.5, 3)$, $(1, 3)$, $(-3, 2)$, $(-1.5, 3)$, $(3, 3)$. The dashed lines represent the restrictions of Eqs. (9).

Lemma 5. *Only one of two situations is possible:*

(1) none of the conditions C1-C5 is satisfied;

(2) only one of the following conditions is true: C1-C5.

Proof. First of all, one can see that each of the situations is possible by looking into examples in Fig. 4. Now let us show that the other combinations are not allowed.

C1 and C2; C4 and C5 cannot be simultaneously true. These statements are the consequences of the fact that set $\mathcal{L}_+^n$ contains a point with $Fr_+ = -1$ only if $\mathcal{P}_+ = \varnothing$ (see Fig. 3a, b). Therefore, if $\mathcal{P}_+ = \varnothing$, then C1 and C4 are not true; if $\mathcal{P}_+ \neq \varnothing$, then C2 and C5 are not true, since $Fr_+ > -1$.

C3 and C4; C3 and C5 cannot be simultaneously true. Line $\Phi_{RL}(\mathcal{L}_+^n)$ (with $Fr_- > -1$) can be continuously connected to line $\Psi_{RL}(\mathcal{P}_+)$ or $\Psi_{RL}(\mathcal{L}_+^n)$ in the limit as $c_b^* \to 1$ when 1-shock wave in the right half-plane has zero velocity or when $Fr_+ = -1$, respectively (see, for example, the intersections of these lines in Fig. 4a, b). The point of intersection corresponds to the minimum value of u_- for $\Phi_{RL}(\mathcal{L}_+^n)$ and the maximum value of u_- for $\Psi_{RL}(\mathcal{P}_+)$ or $\Psi_{RL}(\mathcal{L}_+^n)$ (because $c_b^* \to 1$, see Appendix C, Lemma C-2). Hence, the combined line $\Phi_{RL}(\mathcal{L}_+^n) \cup \Psi_{RL}(\mathcal{P}_+)$ or $\Phi_{RL}(\mathcal{L}_+^n) \cup \Psi_{RL}(\mathcal{L}_+^n)$ is an increasing function $u_-(c_-)$ (see Appendix B, Lemma B-4 and Appendix C, Lemma C-4). There can be only one intersection with the decreasing function $\mathcal{L}_-^n$.

Cases C1 or C2 cannot be true together with one of the cases C3-C5. As it has been mentioned before, if $Fr_- > -1$ lines $\Phi_{RL}(\mathcal{L}_+^n)$ and $\Psi_{RL}(\mathcal{P}_+)$ ($\Psi_{RL}(\mathcal{L}_+^n)$) for $Fr_+ < -1$ ($Fr_+ = -1$) can be seen as one line, which can have only one intersection with $\mathcal{L}_-^n$. We need to show that this intersection cannot coexist with cases C1 or C2. Consider the limiting case when lines $\Psi_{RL}(\mathcal{P}_+)$ ($\Psi_{RL}(\mathcal{L}_+^n)$) and $\mathcal{L}_-^n$ no longer intersect, i.e. the leftmost point $c_b^* \to 0$ ($c_b^* = 0$ is excluded from the resonant wave, see Section 4). It is the limiting situation, when the 2-shock wave to the left has a zero velocity, and the point $\Phi_{RL}(\mathcal{P}_+)$ ($\Phi_{RL}(\mathcal{L}_+^n)$) lies on the boundary of domain $\mathcal{S}_-$. That is why, if small change in the initial conditions leads to the disappearance of the intersection between $\Psi_{RL}(\mathcal{P}_+)$ or $\Psi_{RL}(\mathcal{L}_+^n)$ and $\mathcal{L}_-^n$, then $\Phi_{RL}(\mathcal{P}_+) \cap \mathcal{S}_- \neq \varnothing$ or $\Phi_{RL}(\mathcal{L}_+^n) \cap \mathcal{S}_- \neq \varnothing$. □

6. Uniqueness of the solution

To construct a unique solution of the problem we involve an additional requirement: *discharge* $q = u_+ c_+^2 = u_- c_-^2$ *at* $x = 0$ *continuously depends on initial conditions* $\boldsymbol{w}_L$, $\boldsymbol{w}_R$. 'Non-physical' in this sense solutions are discarded by considering the variation of q at the bifurcation points, where such solutions appear. First we show the uniqueness among the solutions with $q > 0$ (Lemma 6), then we consider a general case (Theorem 1).

Lemma 6. *From the condition of continuous dependence of q on $\mathbf{w}_L$ and $\mathbf{w}_R$ it follows that for $q > 0$ only one of conditions B1, B2 or B3 can be satisfied.*

Proof. From Lemma 3 the following types of two-point and three-point intersections are possible: B1, B4 (Fig. 3g); B1, B5 (Fig. 3h); two intermediate cases B1, B2 and B1, B3; B1, B2, B4 (Fig. 3d); B1, B3, B4 (Fig. 3e); B1, B3, B5 (Fig. 3f). Let us show that for a three-point intersection there are bifurcation points with a discontinuous change of q.

Configuration B1, B3, B5 can be continuously connected to the configuration, where only B3 is satisfied (Fig. 3f, c). This transition goes through the limiting point, when $\Phi_{LR}(\mathcal{P}_-)$ belongs to the border of set $\mathcal{S}_+$ and $\Psi_{LR}(\mathcal{P}_-) = \varnothing$ ($c_b^* \to 1$), when 1-shock wave to the right has zero velocity. For this limit $q_{B1} = q_{B5} \neq q_{B3}$ (the subscript denotes the case). Therefore, the transitions from B1 to B3 and from B5 to B3 are not continuous. Hence, *cases B1 and B5 in configuration B1, B3, B5 should be discarded, as well as case B1 should be discarded in intermediate configuration B1, B3.*

Configuration B1, B3, B4, can be connected to B1, B3, B5 through continuous transition from B4 to B5, see Fig. 3e, f. At the limiting point of transition $q_{B1} = q_{B4} = q_{B5} \neq q_{B3}$. Since we have already rejected cases B1, B5 from configuration B1, B3, B5 *cases B1, B4 should be discarded from configuration B1, B3, B4.*

Similar reasoning can be carried out for the connection of B1, B2, B4 to B1, B3, B4 (Fig. 3d, e). It follows that *cases B1, B4 should be discarded from configuration B1, B2, B4.* Since configuration B1, B2 can be considered as the limit of configuration B1, B2, B4 as $\Phi_{LR}(\mathcal{P}_-)$ tends to the border $Fr_+ = 1$ of set $\mathcal{S}_+$, *case B1 should be discarded from intermediate configuration B1, B2.*

Other transitions of these (three-point) configurations is continuous with respect to cases B2 and B3. Hence, for three-point intersections only cases B2 and B3 satisfy the requirement of continuous dependence of q on initial conditions.

Two-point intersections B1, B4 and B1, B5 should be discarded. These cases appear from B1, B3, B4 and B1, B3, B5 when intersection B3 continuously becomes A1, $q = 0$ (case A1 with $c_- = 1$, which is equal to case A2 with $c_+ \to 0$). At this point $q_{B1} = q_{B4} \neq 0$ or $q_{B1} = q_{B5} \neq 0$. Hence, since we rejected cases B1, B4 and B1, B5 in three-point configurations, their appearance (during transition B3-A1) does not satisfy the requirement of continuous dependence of q on initial conditions. □

Theorem 1. *With a given $\mathbf{w}_L$ and $\mathbf{w}_R$, the solution of the Riemann problem with a discontinuous bottom exists and it is unique if q continuously depends on the initial conditions.*

Proof. Previously, in Lemmas 1, 5, 6 it was shown that with given initial conditions $\mathbf{w}_L$ and $\mathbf{w}_R$ there can be only one solution with $q = 0$, one with $q > 0$, and one with $q < 0$. Let us show that a situation when two solutions from different classes ($q < 0$, $q = 0$, $q > 0$) correspond to the same initial conditions $\mathbf{w}_L$, $\mathbf{w}_R$ is impossible.

First we show that the solution with $q = 0$ (A1, A2) cannot coexist with the solution with $q > 0$ or $q < 0$. One can notice that A2 is the limiting case for the existence of the solutions of types B and C (transitions B3-A2, C3-A2). Hence, A2 and any B-, C-cases cannot be true simultaneously.

If A1 is true, then $\mathcal{P}_+ = \varnothing$ and $\mathcal{L}_+ = \varnothing$, because $u_R \geq 2c_R$ or $c_R = 0$. Hence, there are no solutions of type C. If solution A1 corresponds to $c_- = 0$, then $u_L \leq -2c_L$, hence, $\mathcal{P}_- = \varnothing$ and $\mathcal{L}_- = \varnothing$ and there cannot be solutions of type B. Thus, there is one case left: A1 corresponds to some c_-^0 such that $0 < c_-^0 \leq 1$. It can coexist with a two-point intersection B1, B4, which was discarded in Lemma 6. Hence, A1 and any B-, C-cases cannot be true simultaneously.

From the mutual arrangements of the sets (Figs. 3, 4) it can be seen that intersections of types B and C cease to exist only when they are transformed into the intersection of type A. Moreover, one can see that from a solution of type A it is impossible to organize simultaneous transitions to a solution of type B and a solution of type C. Hence, if the solution exists it is unique.

Let us assume that at some initial condition the solution does not exist. One can build the continuous curve in the space of initial conditions to the point, for which the solution exists. Therefore, for some initial condition, the solution of type A, B or C ceases to exist. The consideration of all mutual arrangements of the sets (for example, in Figs. 3, 4) shows that it is impossible. □

7. Examples of all possible configurations

All possible types of flow configuration are shown in Fig. 5. For each plot values $\mathbf{w}_-$ and $\mathbf{w}_+$ are fixed. Hence, any combination of configurations to the left and to the right of the discontinuity is a particular exact solution. Initial conditions for all solutions are given in Table 1, plots of $z(x,t) = b(x) + h(x,t)$ in Fig. 5 correspond to $t = 0.5$. It should be noted that these solutions were obtained in dimensionless variables (see Section 2).

There are 128 types of flow configurations (with $\Delta b > 0$). In this classification we distinguish flow direction on the discontinuity; the Froude number values at $x = +0$, $x = -0$ (critical, sub- and supercritical Froude numbers); appearance

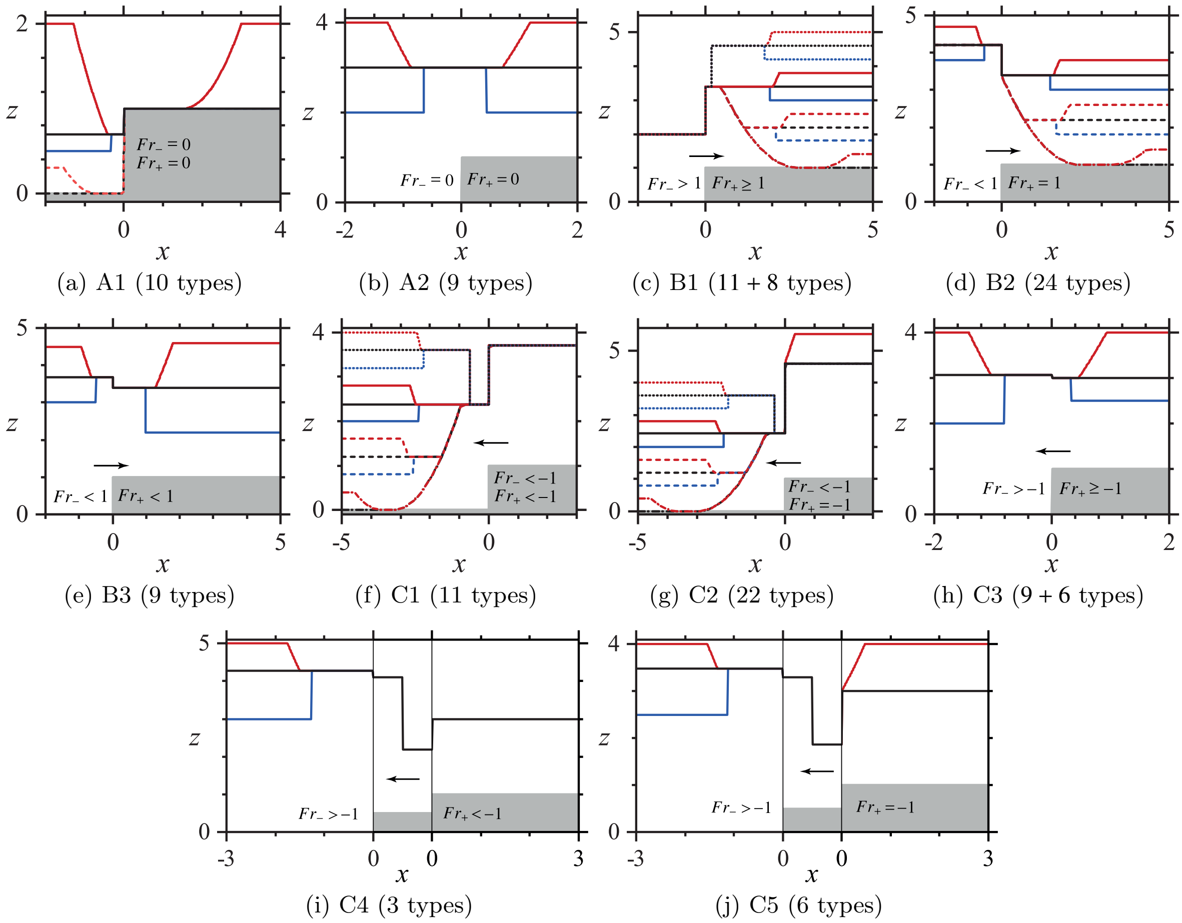


Fig. 5: All possible types of configurations for cases A-C. The shaded area indicates the bottom level: $b = 0$ at $x < 0$ and $b = 1$ at $x > 0$, the resonant waves (i, j) have $b = 0.5$ at $x = 0$ ($c_b^{*2} = 0.5$). The lines show the free surface, $z(x) = b(x) + h(x)$. Each combination of two lines to the left and to the right of bottom discontinuity represents the exact solution of the Riemann problem. The arrows show the flow direction at $x = 0$. The initial conditions for each configuration are given in Table 1, $t = 0.5$.

of shocks, rarefactions and dry bottom regions. For cases B1 and C3 (Fig. 5c, h) $Fr_+ \geq 1$ and $Fr_+ \geq -1$, hence, we should separate situations with $|Fr_+| = 1$. After the separation there will be extra 8 (1-shock wave will not be possible in Fig. 5c) and 6 (2-shock wave will not be possible in Fig. 5h) types of flow. Thus, the amount of all possible configurations with $\Delta b > 0$ is equal to (see Fig. 5)

$$\underbrace{10}_{\text{A1}} + \underbrace{9}_{\text{A2}} + \underbrace{11}_{\text{B1}\,(Fr_+>1)} + \underbrace{8}_{\text{B1}\,(Fr_+=1)} + \underbrace{24}_{\text{B2}} + \underbrace{9}_{\text{B3}} + \underbrace{11}_{\text{C1}} + \underbrace{22}_{\text{C2}} + \underbrace{9}_{\text{C3}\,(Fr_+>-1)} + \underbrace{6}_{\text{C3}\,(Fr_+=-1)} + \underbrace{3}_{\text{C4}} + \underbrace{6}_{\text{C5}} = 128. \tag{23}$$

8. Conclusions

It is shown that the exact solution of the Riemann problem with discontinuous topography is unique under the assumption of the continuous dependence on the initial data of the discharge at the bottom discontinuity. The uniqueness of the solution gives the opportunity for using an exact solution in numerical methods (such as Godunov-type methods), since there is no need to choose between several solutions.

The method used to study the solvability of the Riemann problem with discontinuous bottom is based on the search for pairs (c_-, u_-), (c_+, u_+) from the sets of all possible values to the left and right of the discontinuity, which can be connected by the relations on it. These relations are obtained treating the bottom discontinuity as the limiting case of smoothly changing bottom as the length of the transition interval tends to zero. The 'non-physical' solutions are discarded by tracing the discharge continuity at bifurcation points, when flow regimes are changing. For the implementation of the Riemann solver algorithm one just needs to correctly switch between the solutions at these bifurcation points (the present work shows when exactly it can be done).

Examples of all 128 possible configurations (with $\Delta b > 0$) of exact solution are given and can be used for testing algorithms.

Table 1: The initial conditions for typical test cases shown in Fig. 5.

Case	(h_L, u_L)			(h_R, u_R)		
A1	(0.0, 0.000)	(0.3, -2.500)	(0.5, 0.262)	(0.0, 0.0)	(1.0, 5.0)	
	(0.7, 0.000)	(2.0, -1.155)				
A2	(2.0, 0.646)	(3.0, 0.000)	(4.0, -0.536)	(1.0, -0.866)	(2.0, 0.000)	(3.0, 0.636)
B1	(1.993, 3.012)			(0.0, 0.000)	(0.4, 8.000)	(0.8, 2.999)
				(1.2, 3.408)	(1.6, 3.746)	(2.0, 2.229)
				(2.4, 2.500)	(2.8, 2.748)	(3.2, 1.576)
				(3.6, 1.793)	(4.0, 1.998)	
B2	(3.80, 1.088)	(4.21, 0.883)	(4.70, 0.651)	(0.0, 0.000)	(0.4, 8.000)	(0.8, 2.048)
				(1.2, 2.457)	(1.6, 2.796)	(2.0, 1.278)
				(2.4, 1.549)	(2.8, 1.798)	
B3	(3.000, 1.029)	(3.688, 0.651)	(4.500, 0.249)	(1.2, 0.051)	(2.4, 1.000)	(3.6, 1.696)
C1	(0.000, 0.000)	(0.400, -8.500)	(0.800, -3.895)	(2.7, -3.000)		
	(1.200, -4.304)	(1.600, -4.643)	(2.000, -3.161)			
	(2.373, -3.414)	(2.800, -3.680)	(3.200, -2.471)			
	(3.600, -2.688)	(4.000, -2.894)				
C2	(0.000, 0.000)	(0.400, -8.500)	(0.800, -3.334)	(3.6, -1.897)	(4.5, -1.450)	
	(1.200, -3.742)	(1.600, -4.081)	(2.000, -2.536)			
	(2.422, -2.821)	(2.800, -3.055)	(3.200, -1.911)			
	(3.600, -2.128)	(4.000, -2.334)				
C3	(2.000, 0.363)	(3.072, -0.326)	(4.000, -0.820)	(1.5, -0.882)	(2.0, -0.500)	(3.0, 0.138)
C4	(3.0, -0.256)	(4.276, -0.936)	(5.0, -1.272)	(2.0, -2.000)		
C5	(2.50, -0.239)	(3.48, -0.813)	(4.00, -1.082)	(2.0, -1.414)	(3.0, -0.779)	

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Appendix A. Restrictions on classical solutions, considered in a half-plane

In Section 3, we constructed the sets $\mathcal{D}_+$ ($\mathcal{D}_-$) of all possible values $\boldsymbol{w}_+$ ($\boldsymbol{w}_-$) with a given $\boldsymbol{w}_R$ ($\boldsymbol{w}_L$) using the restrictions, Eqs. (9), which express that shock and rarefaction waves lie in the considered half-plane. In the following we derive these restrictions for the right half-plane $x \geq 0$.

- *Rarefaction waves.* In the case of a 1-rarefaction wave the restriction is $u_+ - c_+ \geq 0$, if $c_+ > c_0$ (it is a necessary condition for the existence of a 1-rarefaction). Similarly, for a 2-rarefaction wave we have $u_0 + c_0 \geq 0$, if $c_0 < c_R$.

- *Shock waves.* The velocity of a 1-shock wave should be no less than 0. Hence, $u_0 c_0^2 - u_+ c_+^2 \geq 0$, if $c_+ < c_0$ (it is a necessary condition for the existence of a 1-shock). Taking into account Eq. (8),

$$u_+ = u_0 + \frac{c_0^2 - c_+^2}{c_0 c_+} \sqrt{\frac{1}{2}(c_0^2 + c_+^2)}, \tag{A.1}$$

  we obtain inequality $u_+ \geq \sigma(c_0, c_+)$, where $\sigma(c_1, c_2) = (c_1/c_2)\sqrt{(c_1^2 + c_2^2)/2}$. Similarly, for a 2-shock wave we have $u_0 \geq -\sigma(c_R, c_0)$, if $c_0 > c_R$.

These restrictions can be rewritten in the form of Eqs. (9).

Appendix B. Properties of mappings $\boldsymbol{\Phi}_{LR}$ and $\boldsymbol{\Phi}_{RL}$

In this appendix we present results on the existence and uniqueness of the mappings Φ_{LR}, Φ_{RL} (Lemmas B-1, B-2) and discuss monotonicity of curves $\Phi_{RL}(\mathcal{L}_+^n)$ and $\Phi_{LR}(\mathcal{L}_-^p)$ corresponding to these mappings (Lemma B-4).

Lemma B-1 (Existence and uniqueness of Φ_{LR})**.** *The condition for the existence of the solution of Eqs.* (15) *with a given* $\boldsymbol{w}_-$ *is* $\varphi(u_-) \geq 3u_*^2/2 + 1$*. If the solution of Eqs.* (15) *with a given* $\boldsymbol{w}_-$ *exists, then it is unique. In addition, if* $|Fr_-| > 1$ *(*$|Fr_-| < 1$*), then* $|Fr_+| \geq 1$ *(*$|Fr_+| \leq 1$*).*

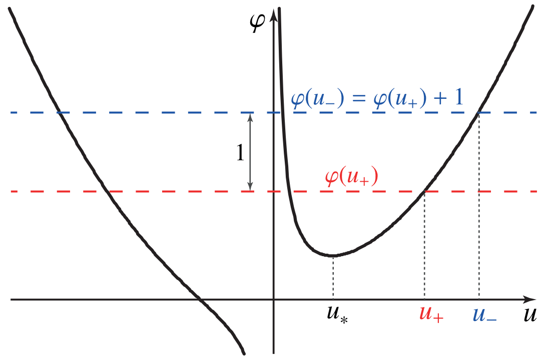

Fig. B.6: Schematic graph of the function $\varphi(u) = u^2/2 + u_*^3/u$ ($u_* > 0$).

Proof. Let $q > 0$ (case $q < 0$ is considered similarly) and u_- be given. We are only interested in $u_+ > 0$ because of $q > 0$. The example of function $\varphi(u)$ graph is shown in Fig. B.6. Depending on u_-, there are zero, one or two solutions of Eqs. (15) with $u_+ > 0$. If $\varphi(u_-) < \varphi(u_*) + 1 = 3u_*^2/2 + 1$, then there are no solutions. If $\varphi(u_-) = \varphi(u_*) + 1$, then there is one solution.

There are two solutions if $\varphi(u_-) > \varphi(u_*) + 1$. One of them can be chosen uniquely using the condition of continuous transition to the solution of the classical Riemann problem when $\Delta b \to 0$ (see, for example, [9]): if $u_- > u_*$ ($u_- < u_*$), then it should be $u_+ > u_*$ ($u_+ < u_*$). Hence, if $Fr_- > 1$ ($Fr_- < 1$), then $Fr_+ \geq 1$ ($Fr_+ \leq 1$), since $Fr_\pm = u_\pm/c_\pm = (u_\pm/u_*)^{3/2}$. Otherwise, in the limit $\Delta b \to 0$ one cannot obtain the solution $u_- = u_+$, because $Fr_- \neq Fr_+$. □

Lemma B-2 (Existence and uniqueness of Φ_{RL})**.** *If $\boldsymbol{w}_+$ is known, then the solution of Eqs.* (15) *always exists.*

- *If $Fr_+ \neq \pm 1$, then the solution is unique. In addition, if $|Fr_+| > 1$ ($|Fr_+| < 1$), then $|Fr_-| > 1$ ($|Fr_-| < 1$).*
- *If $Fr_+ = \pm 1$, then there are two solutions of Eqs.* (15)*: one with $|Fr_-| > 1$ and the other with $|Fr_-| < 1$.*

Proof. Let $q > 0$ (case $q < 0$ is considered similarly) and u_+ be given. From Fig. B.6 it can be seen that there are always two solutions of Eqs. (15) with $u_- > 0$. If $Fr_+ \neq 1$, the solution can be chosen uniquely using the condition of continuous transition to the solution of the classical Riemann problem when $\Delta b \to 0$: if $Fr_+ > 1$ ($Fr_+ < 1$), then $Fr_- > 1$ ($Fr_- < 1$) (see proof of Lemma B-1). If $Fr_+ = 1$ both solutions are possible. □

Lemma B-3. *$\mathcal{L}_+$ ($\mathcal{L}_-$) corresponds to a strictly increasing (decreasing) function $u_+(c_+)$ ($u_-(c_-)$).*

Proof. The function

$$f(c_1, c_2) = \begin{cases} 2(c_1 - c_2), & c_1 \leq c_2, \\ \frac{c_1^2 - c_2^2}{c_1 c_2}\sqrt{\frac{1}{2}(c_1^2 + c_2^2)}, & c_1 > c_2 \end{cases} \tag{B.1}$$

strictly increases (decreases) with respect to the first (second) argument. This can easily be shown by calculating the first derivatives. For instance,

$$0 < \frac{\partial f(c_1, c_2)}{\partial c_1} = \begin{cases} 2, & c_1 \leq c_2, \\ \frac{2c_1^4 + c_1^2 c_2^2 + c_2^4}{c_1^2 c_2 \sqrt{2(c_1^2 + c_2^2)}}, & c_1 > c_2. \end{cases} \tag{B.2}$$

Line $\mathcal{L}_+$ is given by relation $u_+ = u_R + f(c_+, c_R)$, hence it corresponds to a strictly increasing function with respect to argument c_+. Similarly, $\mathcal{L}_-$ corresponds to a strictly decreasing function. □

Lemma B-4. $\Phi_{RL}(\mathcal{L}_+^n)$ *($\Phi_{LR}(\mathcal{L}_-^p)$) corresponds to a strictly increasing (decreasing) function $u_-(c_-)$ ($u_+(c_+)$) provided that $Fr_+ \neq -1$ ($Fr_- \neq 1$).*

Proof. Line $\mathcal{L}_+^n$ is described by the equation $u_+ = u_R + f(c_+, c_R)$, $-c_+ \leq u_+ < 0$. The set $\Phi_{RL}(\mathcal{L}_+^n)$ can be obtained after applying relations

$$\begin{aligned} u_- c_-^2 &= u_+ c_+^2, \\ \frac{1}{2}u_-^2 + c_-^2 &= \frac{1}{2}u_+^2 + c_+^2 + 1. \end{aligned} \tag{B.3}$$

The image of $\Phi_{RL}(\boldsymbol{w}_+)$ exists and is unique, because $Fr_+ \neq -1$ (see Lemma B-2). For each c_+ one can obtain u_+ (on curve $\mathcal{L}_+^n$). Different c_+ gives different q, because $\mathcal{L}_+^n$ is a strictly increasing curve (Lemma B-3). Hence, each c_+ corresponds to a unique pair u_-, c_-. Let us show that $u_-(c_+)$ and $c_-(c_+)$ are strictly increasing functions. We denote the derivative with respect to c_+ by a prime. One can obtain the following estimates for u'_+ from Eq. (B.2)

$$\begin{aligned} &u'_+ = 2, \text{ if } c_+ \le c_R, \\ &2 \le u'_+ \le -\frac{2c_+}{u_+}, \text{ if } c_+ > c_R. \end{aligned} \tag{B.4}$$

Here, we use the relation $u'_+ = \partial f(c_+, c_R)/\partial c_+$ and the restriction $u_+ \ge -\sigma(c_R, c_+)$, if $c_+ > c_R$.

From Eqs. (B.3) expressions for u'_- and c'_- are

$$u'_- = \frac{(c_+^2 - u_+u_-)u'_+ + 2c_+(u_+ - u_-)}{c_-^2 - u_-^2}, \quad c'_- = \frac{(u_+c_-^2 - u_-c_+^2)u'_+ + 2c_+(c_-^2 - u_+u_-)}{2c_-(c_-^2 - u_-^2)}. \tag{B.5}$$

If $c_+ \le c_R$, then $u'_+ = 2$ and

$$u'_- = \frac{2(u_+ + c_+)(c_+ - u_-)}{c_-^2 - u_-^2} > 0, \quad c'_- = \frac{(u_+ + c_+)(c_-^2 - u_-c_+)}{c_-(c_-^2 - u_-^2)} > 0. \tag{B.6}$$

If $c_- > c_R$, then from Eqs. (B.4)

$$u'_- = \frac{2c_+(u_+ + c_+)}{c_-^2 - u_-^2} > 0, \quad c'_- \ge \frac{-u_-c_+(c_+ + u_+)}{c_-(c_-^2 - u_-^2)} > 0. \tag{B.7}$$

Thus $u_-(c_-)$ is a strictly increasing function. Statement of the lemma about function $\Phi_{LR}(\mathcal{L}_-^p)$ can be proved similarly. □

Appendix C. Properties of mappings Ψ_{LR} and Ψ_{RL}

In this appendix we discuss the existence, uniqueness (Lemma C-3) and monotonicity (Lemma C-4) of the mappings Ψ_{LR}, Ψ_{RL}.

Lemma C-1. *For a resonant wave, one of the following conditions holds*

(1) if $q > 0$, then $Fr_-^* > 1$, $Fr_+^* < 1$ and $Fr_- > 1$, $Fr_+ \le 1$;

(2) if $q < 0$, then $Fr_-^* > -1$, $Fr_+^* < -1$ and $Fr_- > -1$, $Fr_+ \le -1$.

Proof. The presence of a shock wave is possible under the condition that either $Fr_-^* > 1$, $Fr_+^* < 1$ and $q > 0$, or $Fr_-^* > -1$, $Fr_+^* < -1$ and $q < 0$. From the first relation in Eqs. (16) it follows that if $|Fr_-^*| < 1$ ($|Fr_-^*| > 1$), then $|Fr_-| < 1$ ($|Fr_-| > 1$). From the second relation in Eqs. (16) it follows that if $|Fr_+^*| < 1$ ($|Fr_+^*| > 1$), then $|Fr_+| \le 1$ ($|Fr_+| \ge 1$). □

Lemma C-2. *For a resonant wave*

(1) if $q > 0$ and u_- are fixed, then $u_+(c_b^*)$ is a strictly decreasing function;

(2) if $q < 0$ and u_+ are fixed, then $u_-(c_b^*)$ is a strictly increasing function.

Proof. Let $q > 0$ and u_- be fixed. $Fr_+ \le 1$ (Lemma C-1), therefore $\varphi(u_+)$ strictly decreases (Fig. B.6). This allows us to study the behaviour of $\varphi(u_+)$ (instead of u_+), depending on the c_b^*. From relation (see Eqs. (16))

$$\varphi(u_+) = \varphi(u_+^*) - \varphi(u_-^*) + \varphi(u_-) - 1 \tag{C.1}$$

it follows that the behaviour of function $u_+(c_b^*)$ is determined by the difference $\varphi(u_+^*) - \varphi(u_-^*)$, since the last two terms do not depend on c_b^*. We represent this difference in the form

$$\varphi(u_+^*) - \varphi(u_-^*) = \chi(u_+^*) - \chi(u_-^*), \tag{C.2}$$

where

$$\chi(u) = \varphi(u) - \psi(u) = \frac{(u^2 - u_*^2)(u - u_*)^2}{2u^2}, \quad \chi'(u) = \frac{(u^3 - u_*^3)(u - u_*)}{u^3}. \tag{C.3}$$

It can be seen that $\chi(u)$ is a strictly increasing function. From the third relation of Eqs. (16) it follows that if u_+^* increases (decreases), then u_-^* decreases (increases). From the first relation of Eqs. (16) it follows that with the growth of c_b^* value u_-^* decreases and, consequently, u_+^* increases. Hence $\chi(u_+^*)$ and $-\chi(u_-^*)$ increase. Therefore $u_+(c_{b^*})$ decreases with the growth of c_b^*.

Part (2) can be proved similarly. □

Lemma C-3. *For a resonant wave*

(1) if $q > 0$ and u_- are fixed, then the solution of Eqs. (16) exists under the condition $\varphi(u_-) > 3u_*^2/2 + 1$ and it is unique for each $0 < c_b^* < 1$;

(2) if $q < 0$ and u_+ are fixed, then there is always a unique solution of Eqs. (16) for each $0 < c_b^* < 1$.

Proof. (1) Let $q > 0$ and u_- be fixed. A resonant wave exists if there exists c_b^* such that $0 < c_b^* < 1$ and $\varphi(u_+) \geq \varphi(u_*)$ (in this case each equation in the system of Eqs. (16) has a solution). The maximum of $\varphi(u_+)$ is attained with the minimal u_+ (because $Fr_+ \leq 1$, see Lemma C-1), i.e. at $c_b^* \to 1$ (see Lemma C-2). So, we consider the limit case $c_b^* \to 1$. Therefore, Eqs. (16) take the form

$$\begin{aligned} \varphi(u_-) &= \varphi(u_-^*) + 1, \\ \psi(u_-^*) &= \psi(u_+). \end{aligned} \tag{C.4}$$

The condition of the existence of the solution with $Fr_-^* > 1$ (see Lemma C-1) for this system is

$$\varphi(u_-) > \tfrac{3}{2}u_*^2 + 1. \tag{C.5}$$

The uniqueness for each c_b^* is a consequence of the fact that Φ_{LR} is single-valued.

(2) Let $q < 0$ and u_+ be fixed. The solution of the first and second relations of Eqs. (16) always exists because they are solved with the known values of u_+ and u_-^* (i.e. here we use function Φ_{RL}, Lemma B-2). If $Fr_+ < -1$ and $Fr_-^* > -1$, then the solution is unique because the function Φ_{RL} is single-valued (Lemma B-2). From Lemma C-1 Fr_-^* is actually greater than -1, however, $Fr_+ = -1$ is also possible. Despite the fact that function Φ_{RL} is not single-valued for $Fr_+ = -1$, we know that $Fr_+^* < -1$ (Lemma C-1) and, therefore, the solution can be chosen uniquely. □

Lemma C-4. *$\boldsymbol{w}_+ = \Psi_{LR}(\boldsymbol{w}_-)$ (with $q > 0$) and $\boldsymbol{w}_- = \Psi_{RL}(\boldsymbol{w}_+)$ (with $q < 0$) correspond to strictly decreasing and strictly increasing functions $u_+ = u_+(c_+)$ and $u_- = u_-(c_-)$.*

Proof. We conduct the justification for Ψ_{LR}, for Ψ_{RL} it is similar. Let $\boldsymbol{w}_-$ be given, then $\Psi_{LR}(\boldsymbol{w}_-)$ defines a parametric curve $u_+ = u_+(c_b^*)$, $c_+ = c_+(c_b^*)$. The discharge at $x = 0$ is constant $u_+c_+^2 = u_-c_-^2 = \text{const}$. Hence, u_+ and c_+ satisfy equation $u_+ = q/c_+^2$. Since $q > 0$, Ψ_{LR} is a strictly decreasing function. □

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